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# Supersymmetry in singular spaces

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## Abstract

We discuss supersymmetry in spaces with a boundary, i.e. singular spaces. In particular, we discuss the situation in ten and five dimensions. In both these cases we review the construction of supersymmetric domain wall actions situated at the boundary. These domain walls act as sources inducing a jump in the mass parameter ( $D=10$ ) or gauge coupling constant ( $D=5$ ). Despite these singularities, supersymmetry can be formulated, maintaining its role as a square root of translations in this singular space.

In  $D=10$  we discuss a generalized form of IIA/IIB supergravity depending on *all* R-R potentials  $C^{(p)}$  ( $p = 0, 1, \dots, 9$ ) as the effective field theory of Type IIA/IIB superstring theory. The case of 8-branes is studied in detail using the new bulk & brane action. As an application of our results we derive a quantization of the mass parameter and the cosmological constant in string units.

In  $D=5$  the setup is designed for the application to the Randall–Sundrum (RS) scenario. The possibility of a supersymmetric RS scenario relies on the existence of a special domain-wall solution containing a warp factor with the correct asymptotic behaviour such that gravity is suppressed in the transverse direction. Whether or not such a domain-wall solution exists depends on the detailed properties of the scalar potential.

Finally, we review the conformal approach as a way to construct general matter coupled supergravity theories and hence general scalar potentials.

## 1 Motivation

There are several situations in string theory where we encounter spaces with a boundary, i.e. singular spaces, see Fig. 1. In the figure (which is specialized to  $D=5$ )  $x$  denotes a compactified direction,  $x \simeq x + 2\tilde{x}$ . In the so-called 2-brane scenario there are two branes, one at each boundary, i.e. one at  $x = 0$  and one at  $x = \tilde{x}$ . There is moreover an orbifold condition relating points  $x$  and  $-x$ . Thus, the  $D$ -dimensional manifold ( $D=10$  or  $D=5$ ) has the form  $M_D = M_{D-1} \times S^1/\mathbb{Z}_2$ .

The highest-dimensional example of this situation is the Horava–Witten scenario in  $D=11$  [1]. In this case two ten-dimensional manifolds are embedded in an eleven-dimensional space. The highest-dimensional example in string theory is the Type I' string theory. Here the domain walls are 8-branes. The orbifold direction is the transverse direction of the branes that fill the rest of the spacetime. Now, the orbifold  $S^1/\mathbb{Z}_2$  being a compact space, we cannot place a single charged object in it, but we have to have at least two oppositely charged objects. However, this kind of system cannot be in supersymmetric equilibrium unless their tensions

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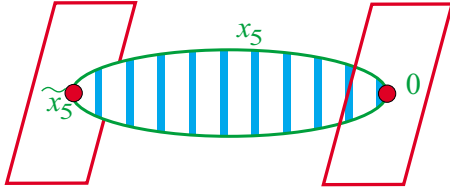


Fig. 1. Two-brane scenario. The  $x$ -direction is a circle with branes at opposite ends and a  $\mathbb{Z}_2$  identification of points symmetric w.r.t.  $x = 0$

also have opposite signs. In Type I' string theory there are negative-tension objects which are called O8-planes. O8-planes can only sit at orbifold points because they require the spacetime to be mirror symmetric in their transverse direction and, thus, they are positioned at the two endpoints of the segment  $S^1/\mathbb{Z}_2$ . We have now a negative tension brane at one endpoint and a positive tension (opposite R-R charge) brane at the other endpoint. The positive tension brane can be identified as a combination of an O8-plane and D8-branes with positive total tension and the negative tension brane as a combination of an O8-plane and D8-branes with negative total tension.

From a phenomenological point of view the case  $D=5$  is particularly interesting in view of the possibility of a supersymmetric Randall-Sundrum (RS) scenario. In this case we have a 3-brane in a 5-dimensional bulk.

It is not obvious how supersymmetry can be implemented in a space with domain walls. The wall is at a fixed place and its presence seems to lead to a breaking of translations orthogonal to the plane. Supersymmetry, being the square root of translations, seems rather difficult to realize in this context. In this lecture we will report on our work [2] and show how this obstacle can be avoided.

First we will remind other results in the literature. In the  $D=11$  Horava-Witten scenario supersymmetry is obtained by a cancellation between anomalies of the bulk theory and a non-invariance of the classical brane action. Lukas, Ovrut, Stelle and Waldram [3,4] reduced this on a Calabi-Yau manifold to five dimensions, and further developed this setup in five dimensions.

Following the work of Randall-Sundrum there has been a revival of interest in the  $D=5$  situation. As far as supersymmetry is concerned there have been two approaches. In the first approach the gauge coupling constant does not jump when one crosses the domain wall [5,6]. In the second one the gauge coupling constant does make a jump [7,8]. Our approach follows the second scenario [2]. Recently, we have shown that the techniques we developed in  $D=5$  also work in  $D=10$  except that the gauge coupling constant  $g$  must be replaced by a mass parameter  $m$  [9]. We first discuss the implementation of supersymmetry in a ten-dimensional singular space.

## 2 D = 10

We will be mainly interested in massive IIA supergravity [10]. This theory is based on the superalgebra (omitting central charges):

$$\{Q_\alpha^\pm, Q_\beta^\pm\} = (P^\pm \gamma^\mu C^{-1})_{\alpha\beta} P_\mu. \quad (1)$$

The basic supergravity multiplet representing this algebra is obtained by multiplying a left-handed and right-handed vector multiplet, i.e.

$$(8_v + 8_s)_L \otimes (8_v + 8_c)_R = \underbrace{35_v + 28 + 1}_{\text{NS-NS}} + \underbrace{8_v + 56_v}_{\text{R-R}} \quad (2)$$

$$+ \underbrace{8_c + 56_s}_{\text{NS-R}} + \underbrace{8_s + 56_c}_{\text{R-NS}} = 128 + 128. \quad (3)$$

The field content of this multiplet is given by

$$\text{IIA} : \left\{ g_{\mu\nu}, B_{\mu\nu}, \phi; C^{(1)}, C^{(3)}, m, \psi_\mu^\pm, \lambda^\pm \right\}, \quad (4)$$

where  $m$  is a mass parameter. The first three fields are the NS-NS fields. The 1-form gauge field  $C^{(1)}$  and three-form gauge field  $C^{(3)}$  are R-R fields. The gravitini  $\psi_\mu^\pm$  and dilatini  $\lambda^\pm$  are (non-chiral) Majorana spinors. It turns out that one can realize the  $N=2$  supersymmetry on the R-R gauge fields of higher rank as well. These are usually incorporated via duality relations (see below). Note that the mass parameter  $m$  describes a deformation of the massless ( $m = 0$ ) case introducing a cosmological constant proportional to  $m^2$ .

To treat the R-R potentials democratically we propose a new *democratic formulation* which describes the dynamics of the bulk supergravity in the most elegant way. It is the formulation that naturally describes the coupling of the bulk supergravity to the D-branes [11].

It turns out that the democratic formulation, although very elegant, is not well suited for the purpose of adding brane actions at the boundary of the space. We therefore give at a later stage a different so-called *dual formulation* where the constant mass parameter  $m$  has been replaced by a field.

### The democratic formulation

It is well-known that in ten dimensions a  $p$ -form gauge field  $C^{(p)}$  is dual to a  $(8-p)$ -form gauge field  $C^{(8-p)}$ . Including IIB supergravity this leads to the following equivalences:

$$\begin{array}{ll} C^{(1)} \sim C^{(7)} & C^{(0)} \sim C^{(8)} \\ \text{IIA} : C^{(3)} \sim C^{(5)} & \text{IIB} : C^{(2)} \sim C^{(6)} \\ m \sim C^{(9)} & C^{(4)} \sim C^{(4)} \\ & C^{(10)} \end{array} \quad (5)$$

In the IIB case the 4-form potential  $C^{(4)}$  is self-dual. The IIB ten-form potential  $C^{(10)}$  was introduced in [12]. Note that the highest potential  $C^{(9)}$  in IIA does not carry degrees of freedom. This 9-form potential can be seen as the potential dual to the constant mass parameter  $m$ .

In the democratic formulation all potentials and dual potentials are treated equivalently. To explicitly introduce the democracy among the R-R potentials we propose to work with the potentials and dual potentials at the same time. Of course this enlarges the number of degrees of freedom. Since a  $p$ - and an  $(8-p)$ -form potential carry the same number of degrees of

freedom, the introduction of the dual potentials doubles the R-R sector. This doubling of number of degrees of freedom will be taken care of by a constraint, relating the lower- and higher-rank potentials (see below). This new formulation of supersymmetry is inspired by the bosonic construction of [11]. We will discuss the democratic formulation of the IIA and IIB case simultaneously.

The extended field content in the democratic formulation is given by

$$\begin{aligned} \text{IIA} : \quad & \left\{ g_{\mu\nu}, B_{\mu\nu}, \phi, C_{\mu}^{(1)}, C_{\mu\nu\rho}^{(3)}, C_{\mu\cdots\rho}^{(5)}, C_{\mu\cdots\rho}^{(7)}, C_{\mu\cdots\rho}^{(9)}, \psi_{\mu}, \lambda \right\}, \\ \text{IIB} : \quad & \left\{ g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(0)}, C_{\mu\nu}^{(2)}, C_{\mu\cdots\rho}^{(4)}, C_{\mu\cdots\rho}^{(6)}, C_{\mu\cdots\rho}^{(8)}, \psi_{\mu}, \lambda \right\}. \end{aligned} \quad (6)$$

It is understood that in the IIA case the fermions contain both chiralities, while in the IIB case they satisfy

$$\Gamma_{11}\psi_{\mu} = \psi_{\mu}, \quad \Gamma_{11}\lambda = -\lambda, \quad (\text{IIB}). \quad (7)$$

In that case they are doublets, and we suppress the corresponding index.

For notational convenience we group all potentials and field strengths in the formal sums<sup>1</sup>

$$\mathbf{G} = \sum_{n=0,1/2}^{5,9/2} G^{(2n)}, \quad \mathbf{C} = \sum_{n=1,1/2}^{5,9/2} C^{(2n-1)}. \quad (8)$$

It is understood that the above summation is over integers ( $n = 0, 1, \dots, 5$ ) in the IIA case and over half-integers ( $n = 1/2, 3/2, \dots, 9/2$ ) in the IIB case. In the summation range we will always first indicate the lowest value for the IIA case, before the one for the IIB case. Using this notation the bosonic field strengths are given by

$$H = dB, \quad \mathbf{G} = d\mathbf{C} - dB \wedge \mathbf{C} + G^{(0)}\mathbf{e}^B, \quad (9)$$

where it is understood that each equation involves only one term from the formal sums (8) (only the relevant combinations are extracted). The corresponding Bianchi identities then read

$$dH = 0, \quad d\mathbf{G} - H \wedge \mathbf{G} = 0. \quad (10)$$

In this subsection  $G^{(0)} = m$  indicates the constant mass parameter of IIA supergravity. In the IIB theory all equations should be read with vanishing  $G^{(0)}$ .

Due to the appearance of all R-R potentials, the number of degrees of freedom in the R-R sector has been doubled. To get the correct number of degrees of freedom, we must relate the different potentials. We therefore by hand impose the following duality relations

$$G^{(2n)} + \Psi^{(2n)} = (-)^{\text{Int}[n]} \star G^{(10-2n)}, \quad (11)$$

with the fermion bilinears  $\Psi^{(2n)}$  given by

$$\begin{aligned} \Psi_{\mu_1 \cdots \mu_{2n}}^{(2n)} = & \frac{1}{2} e^{-\phi} \bar{\psi}_{\alpha} \Gamma^{[\alpha} \Gamma_{\mu_1 \cdots \mu_{2n}} \Gamma^{\beta]} \mathcal{P}_n \psi_{\beta} + \frac{1}{2} e^{-\phi} \bar{\lambda} \Gamma_{\mu_1 \cdots \mu_{2n}} \Gamma^{\beta} \mathcal{P}_n \psi_{\beta} + \\ & - \frac{1}{4} e^{-\phi} \bar{\lambda} \Gamma_{[\mu_1 \cdots \mu_{2n-1}} \mathcal{P}_n \Gamma_{\mu_{2n}] \lambda}. \end{aligned} \quad (12)$$

<sup>1</sup> We use the notation and conventions of [9]

We have used the following definitions:

$$\begin{aligned}\mathcal{P} &= \Gamma_{11} \quad (\text{IIA}) \quad \text{or} \quad -\sigma^3 \quad (\text{IIB}), \\ \mathcal{P}_n &= (\Gamma_{11})^n \quad (\text{IIA}) \quad \text{or} \\ &\quad \sigma^1 \quad (n+1/2 \text{ even}), \quad i\sigma^2 \quad (n+1/2 \text{ odd}) \quad (\text{IIB}).\end{aligned}\quad (13)$$

Note that the fermion bilinears satisfy

$$\Psi^{(2n)} = (-)^{\text{Int}[n]+1} \star \Psi^{(10-2n)}.\quad (14)$$

The supersymmetry transformations of the fields (modulo cubic fermion terms) are given by

$$\begin{aligned}\delta e_\mu{}^a &= \bar{\epsilon} \Gamma^a \psi_\mu, \\ \delta \psi_\mu &= \left( \partial_\mu + \frac{1}{4} \phi_\mu + \frac{1}{8} \mathcal{P} H_\mu \right) \epsilon + \frac{1}{16} e^\phi \sum_{n=0,1/2}^{5,9/2} \frac{1}{(2n)!} \mathcal{G}^{(2n)} \Gamma_\mu \mathcal{P}_n \epsilon, \\ \delta B_{\mu\nu} &= -2 \bar{\epsilon} \Gamma_{[\mu} \mathcal{P} \psi_{\nu]}, \\ \delta C_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)} &= -e^{-\phi} \bar{\epsilon} \Gamma_{[\mu_1 \dots \mu_{2n-2}} \mathcal{P}_n \left( (2n-1) \psi_{\mu_{2n-1}} - \frac{1}{2} \Gamma_{\mu_{2n-1}} \lambda \right) + \\ &\quad + (n-1)(2n-1) C_{[\mu_1 \dots \mu_{2n-3}}^{(2n-3)} \delta_\epsilon B_{\mu_{2n-2} \mu_{2n-1}}], \\ \delta \lambda &= \left( \not{\partial} \phi + \frac{1}{12} H \mathcal{P} \right) \epsilon + \frac{1}{8} e^\phi \sum_{n=0,1/2}^{5,9/2} (-)^{2n} \frac{5-2n}{(2n)!} \mathcal{G}^{(2n)} \mathcal{P}_n \epsilon, \\ \delta \phi &= \frac{1}{2} \bar{\epsilon} \lambda,\end{aligned}\quad (15)$$

where  $\epsilon$  is a spinor similar to  $\psi_\mu$ , i.e. in IIB:  $\Gamma_{11}\epsilon = \epsilon$ . Note that for  $n$  half-integer (the IIB case) these supersymmetry rules exactly reproduce the rules given in eq. (1.1) of [12]. The superspace formulation of the above supersymmetry rules have been given in the context of the kappa-symmetry of super Dp-branes [13, 14].

We now describe, in three steps, how to introduce domain wall brane actions.

### Step 1: $\mathbb{Z}_2$ -symmetries

To construct a domain-wall with a mass and a charge we need to (i) replace  $G^{(0)} = m$  by a field  $G^{(0)}(x)$  so that it can jump when crossing the domain wall and (ii) introduce a 9-form potential  $C^{(9)}$  that can couple to the domain wall describing its charge. In other words, we need to replace

$$m \longrightarrow G^{(0)}(x) \quad \text{plus} \quad C^{(9)}.\quad (17)$$

To do this we need a  $\mathbb{Z}_2$ -symmetry under which (i)  $G^{(0)}(x)$  is odd, consistent with its changing sign when crossing the brane, and (ii)  $C^{(9)}$  is even so that it can be coupled to the domain wall. In other words, we require:

$$G^{(0)}(x) : \text{ odd} \qquad C^{(9)} : \text{ even}.\quad (18)$$

In  $D=10$  all branes are democratic. It is therefore natural to consider also the other p-branes. It turns out that for each p a specific  $\mathbb{Z}_2$ -symmetry, called  $\mathbb{Z}_2(p)$ , is needed for adding p-branes. It should contain parity transformations in the transverse directions and it should satisfy

$$G^{(0)}(x) : \text{ odd} \qquad C^{(p+1)} : \text{ even} \qquad (19)$$

Below we first list the different  $\mathbb{Z}_2$  symmetries that are present in Type IIA/IIB superstring theory.

- (1) In the IIB case there is a worldsheet parity symmetry  $\Omega$ . In terms of the Green-Schwarz lightcone action

$$S_{\text{IIB}} = \int d^2\xi \{ \partial_+ X^i \partial_- X^i - i S^a \partial_- S^a - i \tilde{S}^a \partial_+ \tilde{S}^a \} \qquad (20)$$

it is given by

$$\Omega : \sigma \rightarrow -\sigma, \qquad S^a \leftrightarrow \tilde{S}^a. \qquad (21)$$

The IIB supergravity fields transform under  $\Omega$  according to

$$\begin{aligned} \{ \phi, g_{\mu\nu}, B_{\mu\nu} \} &\rightarrow \{ \phi, g_{\mu\nu}, -B_{\mu\nu} \}, \\ \{ C_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)} \} &\rightarrow (-)^{n+1/2} \{ C_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)} \}, \\ \{ \psi_\mu, \lambda, \epsilon \} &\rightarrow \sigma^1 \{ \psi_\mu, \lambda, \epsilon \}, \end{aligned} \qquad (22)$$

where  $\sigma^1$  is a Pauli matrix.

We note that the  $SO(32)$  Type I' superstring is obtained by dividing out the  $N=2$  Type II superstring by  $\Omega$ .

- (2) In the massless IIA case, i.e.  $m = 0$ , there is a similar  $I_9\Omega$ -symmetry involving an additional parity transformation in the 9-direction. This can be understood from T-duality. Writing

$$\begin{aligned} \Omega : \quad X_L &\leftrightarrow X_R, \\ T_9 : \quad X_L^9 &\rightarrow -X_L^9, \end{aligned} \qquad (23)$$

we obtain

$$T_9 \Omega T_9^{-1} = I_9 \Omega, \qquad (24)$$

with  $I_9$  an inversion of  $X^9$ :

$$I_9 : \quad X_L^9 \rightarrow -X_L^9, \qquad X_R^9 \rightarrow -X_R^9. \qquad (25)$$

Writing  $\mu = (\underline{\mu}, \dot{9})$ , the parity of the IIA supergravity fields are given by

$$\begin{aligned}
x^{\dot{9}} &\rightarrow -x^{\dot{9}}, \\
\{\phi, g_{\mu\nu}, B_{\mu\nu}\} &\rightarrow \{\phi, g_{\mu\nu}, -B_{\mu\nu}\}, \\
\{C_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)}\} &\rightarrow (-)^{n+1} \{C_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)}\}, \\
\{\psi_{\underline{\mu}}, \lambda, \epsilon\} &\rightarrow +\Gamma^9 \{\psi_{\underline{\mu}}, -\lambda, \epsilon\}.
\end{aligned} \tag{26}$$

The parity of the fields with one or more indices in the  $\dot{9}$ -direction is given by the rule that every index in the  $\dot{9}$ -direction gives an extra minus sign compared to the above rules.

The  $SO(16) \times SO(16)$  Type I' superstring is obtained by dividing the N=2 IIA superstring by  $I_9\Omega$ .

- (2) For both massless IIA, with  $m = 0$ , and IIB there is a fermion number symmetry  $(-)^{F_L}$ . In terms of the Green-Schwarz action (20) this symmetry is given by

$$\text{IIA} : S^{\dot{a}} \rightarrow -S^{\dot{a}}, \tag{27}$$

$$\text{IIB} : S^a \rightarrow -S^a \tag{28}$$

The action on the supergravity fields is given by

$$\begin{aligned}
\{\phi, g_{\mu\nu}, B_{\mu\nu}\} &\rightarrow \{\phi, g_{\mu\nu}, B_{\mu\nu}\}, \\
\{C_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)}\} &\rightarrow -\{C_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)}\}, \\
\{\psi_{\underline{\mu}}, \lambda, \epsilon\} &\rightarrow +\mathcal{P}\{\psi_{\underline{\mu}}, -\lambda, \epsilon\}, \text{ (IIA)}, \{\psi_{\underline{\mu}}, -\lambda, \epsilon\}, \text{ (IIB)}, \\
\{\psi_{\underline{\mu}}, \lambda, \epsilon\} &\rightarrow +\mathcal{P}\{\psi_{\underline{\mu}}, \lambda, \epsilon\}, \text{ (IIB)}.
\end{aligned} \tag{29}$$

This concludes our description of the different  $\mathbb{Z}_2$ -symmetries. For the 8-brane the  $I_9\Omega$  symmetry satisfies the desired properties (18). For the other  $p$ -branes, it would seem natural to use the  $\mathbb{Z}_2$ -symmetry

$$I_{9,8,\dots,p+1}\Omega \equiv (I_9\Omega)(I_8\Omega) \cdots (I_{p+1}\Omega), \tag{30}$$

where  $I_q\Omega$  is the transformation (26) with 9 replaced by  $q$ , and  $I_q$  and  $\Omega$  commute. However, for some  $p$ -branes ( $p = 2, 3, 6, 7$ ) the corresponding  $C^{(p+1)}$  R-R-potential is odd under this  $\mathbb{Z}_2$ -symmetry. To obtain the correct parity one must include an extra  $(-)^{F_L}$  transformation in these cases, which also follows from T-duality [15]. This leads, for each  $p$ -brane, to the  $\mathbb{Z}_2$ -symmetry indicated in Table 1.

Thus the correct  $\mathbb{Z}_2$ -symmetry for a general IIA  $Op$ -plane is given by

$$\mathbb{Z}_2(p) = ((-)^{F_L})^{p/2} I_{9,8,\dots,p+1}\Omega. \tag{31}$$



Table 1. The  $\mathbb{Z}_2$ -symmetries used in the orientifold construction of an  $Op$ -plane. The T-duality transformation from IIA to IIB in the lower dimension induces each time a  $(-)^{F_L}$

$p$	IIB	IIA
9	$\Omega$	-
8	-	$I_9\Omega$
7	$(-)^{F_L} I_{9,8}\Omega$	-
6	-	$(-)^{F_L} I_{9,8,7}\Omega$
5	$I_{9,8,\dots,6}\Omega$	-
4	-	$I_{9,8,\dots,5}\Omega$
3	$(-)^{F_L} I_{9,8,\dots,4}\Omega$	-
2	-	$(-)^{F_L} I_{9,8,\dots,3}\Omega$
1	$I_{9,8,\dots,2}\Omega$	-
0	-	$I_{9,8,\dots,1}\Omega$

The effect of this  $\mathbb{Z}_2$ -symmetry on the bulk fields reads (the underlined indices refer to the worldvolume directions, i.e.  $\mu = (\underline{\mu}, p+1, \dots, 9)$ )

$$\begin{aligned}
 \{x^{p+1}, \dots, x^9\} &\rightarrow -\{x^{p+1}, \dots, x^9\}, \\
 \{\phi, g_{\mu\nu}, B_{\underline{\mu}\underline{\nu}}\} &\rightarrow \{\phi, g_{\mu\nu}, -B_{\underline{\mu}\underline{\nu}}\}, \\
 \{C_{\underline{\mu}_1 \dots \underline{\mu}_5}^{(5)}, C_{\underline{\mu}_1 \dots \underline{\mu}_9}^{(9)}\} &\rightarrow (-)^{\frac{p}{2}} \{C_{\underline{\mu}_1 \dots \underline{\mu}_5}^{(5)}, C_{\underline{\mu}_1 \dots \underline{\mu}_9}^{(9)}\}, \\
 C_{\underline{\mu}_1 \dots \underline{\mu}_7}^{(7)} &\rightarrow (-)^{\frac{p}{2}+1} C_{\underline{\mu}_1 \dots \underline{\mu}_7}^{(7)}, \\
 \{\psi_{\underline{\mu}}, \epsilon\} &\rightarrow -\alpha \Gamma^{p+1 \dots 9} (-\Gamma_{11})^{\frac{p}{2}} \{\psi_{\underline{\mu}}, \epsilon\}, \\
 \{\lambda\} &\rightarrow +\alpha \Gamma^{p+1 \dots 9} (+\Gamma_{11})^{\frac{p}{2}} \{\lambda\},
 \end{aligned} \tag{32}$$

and for fields with other indices there is an extra minus sign for each replacement of a world-volume index  $\underline{\mu}$  by an index in a transverse direction. We have left open the possibility of combining the symmetry with the sign change of all fermions. This possibility introduces a number  $\alpha = \pm 1$  in the above rules. This symmetry will be used for the orientifold construction.

Having discussed the correct  $\mathbb{Z}_2$ -symmetry to place p-branes at the boundary of the space we next continue with describing the new dual formulation which is most appropriate to describe the coupling of the bulk supergravity to the p-branes.

## Step 2: the dual formulation

We will present here the new dual formulation with action, available for the IIA case only. It is this formulation that we will apply in our construction of the bulk & brane system. The

dual formulation has the desired property that  $m$  is replaced by a field  $G^{(0)}(x)$  which is odd under the  $\mathbb{Z}_2(8) = I_9\Omega$ -symmetry.

The independent fields in the dual formulation are

$$\left\{ e_\mu^a, B_{\mu\nu}, \phi, \underbrace{G^{(0)}, G_{\mu\nu}^{(2)}, G_{\mu_1\cdots\mu_4}^{(4)}}_{\text{black boxes}}, \underbrace{A_{\mu_1\cdots\mu_5}^{(5)}, A_{\mu_1\cdots\mu_7}^{(7)}, A_{\mu_1\cdots\mu_9}^{(9)}}_{\text{Lagrange multipliers}}, \psi_\mu, \lambda \right\}. \quad (33)$$

Note that the fields  $m, A^{(1)}, A^{(3)}$  are absent. They will be introduced in a second stage when the equations of motion of the Lagrange multipliers are used to solve for the black boxes. The bulk action reads

$$\begin{aligned} S_{\text{bulk}} = & -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi} [R(\omega(e)) - 4(\partial\phi)^2 + \frac{1}{2}H \cdot H + \right. \\ & - 2\partial^\mu \phi \chi_\mu^{(1)} + H \cdot \chi^{(3)} + 2\bar{\psi}_\mu \Gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho - 2\bar{\lambda} \Gamma^\mu \nabla_\mu \lambda + \\ & + 4\bar{\lambda} \Gamma^{\mu\nu} \nabla_\mu \psi_\nu] + \sum_{n=0,1,2} \frac{1}{2} G^{(2n)} \cdot G^{(2n)} + G^{(2n)} \cdot \Psi^{(2n)} + \\ & - \star \left[ \frac{1}{2} G^{(4)} G^{(4)} B - \frac{1}{2} G^{(2)} G^{(4)} B^2 + \frac{1}{6} G^{(2)2} B^3 + \frac{1}{6} G^{(0)} G^{(4)} B^3 + \right. \\ & \left. - \frac{1}{8} G^{(0)} G^{(2)} B^4 + \frac{1}{40} G^{(0)2} B^5 + e^{-B} \mathbf{G} d(A^{(5)} - A^{(7)} + A^{(9)}) \right] \Big\} + \\ & + \text{quartic fermionic terms}, \end{aligned} \quad (34)$$

where all  $\wedge$ 's have been omitted in the last two lines. In the last term a projection on the 10-form is understood. Here  $\mathbf{G}$  is defined as in (8) but where  $G^{(0)}, G^{(2)}$  and  $G^{(4)}$  are now independent fields (which we will call black boxes) and are no longer given by (9). Note that their Bianchi identities are imposed by the Lagrange multipliers  $A^{(9)}, A^{(7)}$  and  $A^{(5)}$ . The NS-NS three-form field strength is given by (9). Note that the standard action for IIA supergravity can be obtained by integrating out the dual potentials in (34).

The symmetries of the action are similar to those of the democratic formulation with some small changes. In the supersymmetry transformations of the gravitino and gaugino, the sums now extend only over  $n = 0, 1, 2$ :

$$\begin{aligned} \delta_\epsilon e_\mu^a &= \bar{\epsilon} \Gamma^a \psi_\mu, \\ \delta_\epsilon \psi_\mu &= \left( \partial_\mu + \frac{1}{4} \not{\omega}_\mu + \frac{1}{8} \Gamma_{11} \not{H}_\mu \right) \epsilon + \frac{1}{8} e^\phi \sum_{n=0,1,2} \frac{1}{(2n)!} \not{G}^{(2n)} \Gamma_\mu (\Gamma_{11})^n \epsilon, \\ \delta_\epsilon B_{\mu\nu} &= -2 \bar{\epsilon} \Gamma_{[\mu} \Gamma_{11} \psi_{\nu]}, \\ \delta_\epsilon \lambda &= \left( \not{\partial} \phi - \frac{1}{12} \Gamma_{11} \not{H} \right) \epsilon + \frac{1}{4} e^\phi \sum_{n=0,1,2} \frac{5-2n}{(2n)!} \not{G}^{(2n)} (\Gamma_{11})^n \epsilon, \\ \delta_\epsilon \phi &= \frac{1}{2} \bar{\epsilon} \lambda, \\ \delta_\epsilon \mathbf{A} &= e^{-B} \wedge \mathbf{E}, \\ \delta_\epsilon \mathbf{G} &= d\mathbf{E} + \mathbf{G} \wedge \delta_\epsilon B - H \wedge \mathbf{E}, \end{aligned} \quad (35)$$

with

$$E_{\mu_1\cdots\mu_{2n-1}}^{(2n-1)} \equiv -e^{-\phi} \bar{\epsilon} \Gamma_{[\mu_1\cdots\mu_{2n-2}} (\Gamma_{11})^n \left( (2n-1) \psi_{\mu_{2n-1}] - \frac{1}{2} \Gamma_{\mu_{2n-1}] \lambda \right).$$

The transformation of the black boxes  $\mathbf{G}$  follow from the requirement that  $e^{-B}\mathbf{G}$  transforms in a total derivative. Here the formal sums

$$\mathbf{A} = \sum_{n=1}^5 A^{(2n-1)}, \quad \mathbf{E} = \sum_{n=1}^5 E^{(2n-1)}, \quad \mathbf{G} = \sum_{n=0}^5 G^{(2n)}, \quad (36)$$

have been used. Note that the first formal sum in (36) contains fields,  $A^{(1)}$  and  $A^{(3)}$ , that do not occur in the action. The same applies to  $\mathbf{G}$ , which contains the extra fields  $G^{(6)}$ ,  $G^{(8)}$  and  $G^{(10)}$ . Although these fields do not occur in the action, one can nevertheless show that the supersymmetry algebra is realized on them. To do so one must use the supersymmetry rules of (35) and the equations of motion that follow from the action (34).

The gauge symmetries with parameters  $\Lambda$  and  $\Lambda^{(2n)}$  are

$$\begin{aligned} \delta_\Lambda B &= d\Lambda, & \delta_\Lambda \mathbf{A} &= d\mathbf{L} - G^{(0)}\Lambda - d\Lambda \wedge \mathbf{A}, \\ \delta_\Lambda \mathbf{G} &= d\Lambda \wedge (\mathbf{G} - e^B \wedge (d\mathbf{A} + G^{(0)})) + e^B \wedge \Lambda \wedge dG^{(0)}. \end{aligned} \quad (37)$$

Note that, with respect to the R-R gauge symmetry, the  $\mathbf{A}$  potentials transform as a total derivative while the black boxes are invariant. This shows that the  $\mathbf{A}$  potentials describe those combinations of R-R  $\mathbf{C}$  potentials that form the D-brane WZ terms. In fact, the  $\mathbf{A}$ -basis and  $\mathbf{C}$ -basis of R-R potentials are related via the formula  $\mathbf{A} = \mathbf{C} \wedge e^{-B}$  [9].

Finally, there are  $\mathbb{Z}_2$ -symmetries,  $(-)^{F_L}$  and  $I_9\Omega$ , which leave the action invariant. In contrast to the democratic formulation these two  $\mathbb{Z}_2$ -symmetries are valid symmetries even for  $G^{(0)} \neq 0$ . The  $(-)^{F_L}$ -symmetry is given by

$$\begin{aligned} \{\phi, g_{\mu\nu}, B_{\mu\nu}\} &\rightarrow \{\phi, g_{\mu\nu}, B_{\mu\nu}\}, \\ \{G_{\mu_1 \dots \mu_{2n}}, A_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)}\} &\rightarrow -\{G_{\mu_1 \dots \mu_{2n}}, A_{\mu_1 \dots \mu_{2n-1}}^{(2n-1)}\}, \\ \{\psi_\mu, \lambda, \epsilon\} &\rightarrow +\Gamma_{11}\{\psi_\mu, -\lambda, \epsilon\}, \end{aligned} \quad (38)$$

while the second  $I_9\Omega$ -symmetry reads

$$\begin{aligned} x^9 &\rightarrow -x^9, \\ \{\phi, g_{\underline{\mu}\underline{\nu}}, B_{\underline{\mu}\underline{\nu}}\} &\rightarrow \{\phi, g_{\underline{\mu}\underline{\nu}}, -B_{\underline{\mu}\underline{\nu}}\}, \\ \{G_{\underline{\mu}_1 \dots \underline{\mu}_{2n}}, A_{\underline{\mu}_1 \dots \underline{\mu}_{2n-1}}^{(2n-1)}\} &\rightarrow (-)^{n+1}\{G_{\underline{\mu}_1 \dots \underline{\mu}_{2n}}, A_{\underline{\mu}_1 \dots \underline{\mu}_{2n-1}}^{(2n-1)}\}, \\ \{\psi_{\underline{\mu}}, \lambda, \epsilon\} &\rightarrow +\Gamma^9\{\psi_{\underline{\mu}}, -\lambda, \epsilon\}. \end{aligned} \quad (39)$$

### Step 3: adding brane actions

Having established supersymmetry in the bulk, we now turn to supersymmetry on the brane. For this purpose we choose spacetime to be

$$\mathcal{M}^{p+1} \times T^{9-p}/\mathbb{Z}_2(p), \quad (40)$$

with radii  $R^{\bar{\mu}}$  of the torus that may depend on the world-volume coordinates. All fields satisfy

$$\Phi(x^{\bar{\mu}}) = \Phi(x^{\bar{\mu}} + 2\pi R^{\bar{\mu}}), \quad (41)$$

with  $\bar{\mu} = (p + 1, \dots, 9)$ . The parity symmetry (31) relates the fields in the bulk at  $x^{\bar{\mu}}$  and  $-x^{\bar{\mu}}$ . At the fixed point of the orientifolds, however, this relation is local and projects out half the fields. This means that we are left with only  $N = 1$  supersymmetry at the fixed points, where the branes will be inserted. Consider for example a nine-dimensional orientifold. The projection truncates the bulk  $N = 2$  supersymmetry to a  $N = 1$  supersymmetry on the brane; only half of the 32 components of  $\epsilon$  are even under (32). The original field content, a  $D = 10$ ,  $(128 + 128)$ ,  $N = 2$  supergravity multiplet, gets truncated on the brane to a reducible  $D = 9$ ,  $(64 + 64)$ ,  $N = 1$  theory consisting of a supergravity plus a vector multiplet. One may further restrict to a constant torus. This particular choice of spacetime then projects out a  $N = 1$   $(8+8)$  vector multiplet (containing  $e_{\dot{9}}^9$ ), leaving us with the irreducible  $D = 9$ ,  $(56 + 56)$ ,  $N = 1$  supergravity multiplet. Similar truncations are possible in lower dimensional orientifolds, on which the  $(64 + 64)$   $N = 1$  theory also consists of a number of multiplets.

We now propose the following  $p$ -brane action ( $p = 0, 2, 4, 6, 8$ ):

$$\begin{aligned} \mathcal{L}_p &= -e^{-\phi} \sqrt{-g_{(p+1)}} - \alpha \frac{1}{(p+1)!} \epsilon^{(p+1)} C^{(p+1)}, \\ \text{with } \epsilon^{(p+1)} C^{(p+1)} &\equiv \epsilon_{\underline{\mu_0} \dots \underline{\mu_p}}^{(p+1)} C^{(p+1) \underline{\mu_0} \dots \underline{\mu_p}}, \end{aligned} \quad (42)$$

with  $\epsilon^{(p+1) \underline{\mu_0} \dots \underline{\mu_p}} = \epsilon^{(10) \underline{\mu_0} \dots \underline{\mu_p} p+1 \dots \dot{9}}$ , which follows from  $e_{\underline{\mu}}^{\bar{a}} = 0$  (being odd). Here the underlined indices are  $(p + 1)$ -dimensional and refer to the world-volume. The parameter  $\alpha$  is the same that appears in (32) and takes the values  $\alpha = +1$  for *branes*, which are defined to have tension and charge with the same sign in our conventions, and  $\alpha = -1$  for *anti-branes*, which are defined to have tension and charge of opposite signs. Note that due to the vanishing of  $B$  on the brane the potentials  $C^{(p+1)}$  and  $A^{(p+1)}$  are equal. The  $p$ -brane action can easily be shown to be invariant under the appropriate  $N = 1$  supersymmetry:

$$\delta_{\epsilon} \mathcal{L}_p = -e^{-\phi} \sqrt{-g_{(p+1)}} \bar{\epsilon} \left( 1 - \alpha \Gamma^{p+1 \dots 9} (\Gamma_{11})^{\frac{p}{2}} \right) \Gamma^{\underline{\mu}} \left( \psi_{\underline{\mu}} - \frac{1}{18} \Gamma_{\underline{\mu}} \lambda \right). \quad (43)$$

The above variation vanishes due to the projection under (32) that selects branes or anti-branes depending on the sign of  $\alpha$  (+1 or -1 respectively). In the following discussions we will assume  $\alpha = 1$  but the other case just amounts to replacing branes by anti-branes.

By truncating our theory we are able to construct a brane action that only consists of bosons and yet is separately supersymmetric. Having these at our disposal, we can introduce source terms for the various potentials. In general there are  $2^{9-p}$  fixed points. The compactness of the transverse space implies that the total charge must vanish. Thus the total action will read (we take the special case that all branes are equally distributed over all  $2^{9-p}$  fixed points)

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{bulk}} + k_p \mathcal{L}_p \Delta_p, \\ \text{with } \Delta_p &\equiv (\delta(x^{p+1}) - \delta(x^{p+1} - \pi R^{p+1})) \dots (\delta(x^9) - \delta(x^9 - \pi R^9)), \end{aligned} \quad (44)$$

where the branes at all fixed points have a tension and a charge proportional to  $\pm k_p$ , a parameter of dimension  $1/[\text{length}]^{p+1}$ . Since anti-branes do not satisfy the supersymmetry condition (43), we need both positive and negative tension branes to accomplish vanishing total charge. As explained in the introduction we are going to interpret the negative tension branes as O-planes.

The equations of motion following from (44) induce a  $\delta$ -function in the Bianchi identity of the  $8 - p$ -form field strength. In general, an elegant solution is difficult to find, but in the eight-brane case the situation simplifies. For  $p = 8$  the Bianchi identity reads:

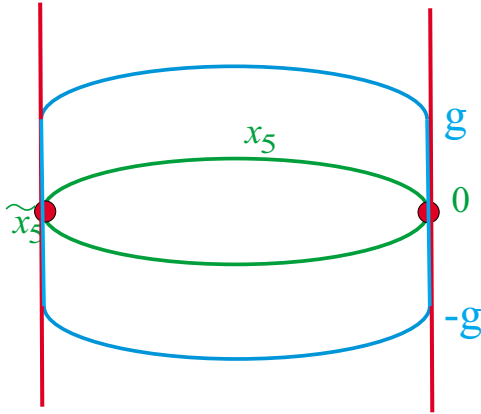


Fig. 2. The mass parameter  $m$  jumps at  $x_9 = 0$  and at  $x_9 = \tilde{x}_9$

$$\frac{1}{\ell_s^8} \partial_9 G^{(0)} = \frac{1}{\ell_s^9} \left( \delta(x^9) - \delta(x^9 - \tilde{x}^9) \right), \quad (45)$$

which leads to the following solution for  $G^{(0)}(x)$ :

$$G^{(0)}(x) \sim \varepsilon(x^9). \quad (46)$$

To summarize, we have constructed a supersymmetric p-brane action, containing only bosons, in a bulk supergravity background with sources at the boundary. The supersymmetry only works for a specific combination of the kinetic and WZ terms on the brane such that the supersymmetry variation is proportional to a projection operator. Since the kinetic term describes the mass and the WZ term the charge of the domain wall we see that supersymmetry is only valid if the mass is equal to the charge, i.e. the domain wall is a 1/2 BPS object. The function  $\varepsilon(x^9)$  jumps as well at  $x^9 = 0$  as at  $x^9 = \tilde{x}^9$ , see Fig. 2 (which refers to the D=5 situation).

It is clear from this picture that we need the second brane. Indeed, one has to come back to the original value of  $m$ , in order that total derivatives in  $x^9$  do not contribute to the action.

As an application of these results we discuss an alternative way of deriving the quantization of the mass parameter  $m$  and, correspondingly, the quantization of the cosmological constant in the bulk.

### Quantization of mass and cosmological constant

Consider the eight-brane case only, i.e. no other branes are present. The precise solution for  $G^{(0)}(x)$ , including the proportionality constant, is given by

$$G^{(0)} = \alpha \frac{n-8}{2\pi\ell_s} \varepsilon(x^9), \quad n \text{ integer}. \quad (47)$$

Thus we may identify the mass parameter of Type IIA supergravity as follows:

$$m = \begin{cases} \alpha \frac{n-8}{2\pi\ell_s}, & x^9 > 0, \\ -\alpha \frac{n-8}{2\pi\ell_s}, & x^9 < 0. \end{cases} \quad (48)$$

The mass is quantized in string units and it is proportional to  $n - 8$  where there are  $2n$  and  $2(16-n)$  D8-branes at each O8-plane. The mass vanishes only in the special case  $n = 8$  when the contribution from the D8-branes cancels exactly the contribution from the O8-planes. In general, the mass takes only the restricted values

$$2\pi\ell_s|m| = 0, 1, 2, 3, 4, 5, 6, 7, 8. \quad (49)$$

This is a quantization of our mass parameter, and for the cosmological constant it follows that

$$m^2 = (G^{(0)})^2 = \left(\frac{n-8}{2\pi\ell_s}\right)^2. \quad (50)$$

Thus the mass parameter and the cosmological constant are quantized in units of the string length in terms of the integers  $n - 8$ .

The quantization of the mass and of the cosmological constant in  $D = 10$  was discussed before in [16–18] as well as in [11, 19]. In the latter two references, two independent derivations of the quantization condition were given. In [19], the T-duality between a 7-brane & 8-brane solution was investigated. Here it was pointed out that, in the presence of a cosmological constant, the relation between the  $D = 10$  IIB R-R scalar  $C^{(0)}$  and the one reduced to  $D = 9$ ,  $c^{(0)}$ , is given via a generalized Scherk–Schwarz prescription:

$$C^{(0)} = c^{(0)}(x^9) + mx^8. \quad (51)$$

Here  $(x^8, x^9)$  parametrize the 2-dimensional space transverse to the 7-brane.  $x^9$  is a radial coordinate whereas  $x^8$  is periodically identified (it corresponds to a U(1) Killing vector field):

$$x^8 \sim x^8 + 1. \quad (52)$$

Furthermore, due to the  $SL(2, \mathbb{Z})$  U-duality, the R-R scalar  $C^{(0)}$  is also periodically identified:

$$C^{(0)} \sim C^{(0)} + 1. \quad (53)$$

Combining the two identifications with the reduction rule for  $C^{(0)}$  leads to a quantization condition for  $m$  of the form

$$m \sim \frac{n}{\ell_s}, \quad n \text{ integer}. \quad (54)$$

The same result was obtained by a different method in [11].

We are able to give a new, and independent, derivation of the quantization condition for the mass and cosmological constant. The conditions given in (48), (50) follow straightforwardly from our construction of the bulk & brane & plane action.

Note that the Scherk–Schwarz reduction in (51) and the quantization of  $SL(2, \mathbb{R})$  were essential in deriving the quantization of  $m$ . In the new dual formulation we can derive a similar T-duality relation between the 7-brane and the 8-brane, including the source terms. However, in this case the T-duality relation does not imply a quantization condition for  $m$  since we do not know how to realize the  $SL(2, \mathbb{R})$  symmetry in the dual formulation. Another noteworthy feature is that the derivation of the T-duality rules in the dual formulation does not require a Scherk–Schwarz reduction. This is possible due to the fact that the R-R scalar only appears after solving the equations of motion.

To conclude the D=10 case, we have constructed a  $\delta$ -function modification of IIA/IIB supergravity which is an interesting result in itself. There are several further issues, for instance

- How do we add matter on the brane in a supersymmetric way?
- We have established D=10 (linear) supersymmetry with sources. Can we do the same for a  $\kappa$ -symmetric Green-Schwarz action?
- Can we relate the D=10 results described in this lecture to the Horava-Witten scenario in D=11? Note that the “end of the world branes” of Horava-Witten do not carry a charge.

We hope to address some of the above issues in the not too distant future. In the meantime we now turn our attention to the D=5 case.

### 3 D = 5

Like in D=10, the construction of the brane action will involve three steps. In step 1 we concentrate on the bulk D=5 matter coupled supergravity and investigate its  $\mathbb{Z}_2$ -symmetries. We consider the set-up of Fig. 1 where two domain walls or 3-branes have been put at the boundary of the singular space. A quite general D=5 matter coupled supergravity has been given [20] but it may not be excluded that further generalizations are possible [21]. We will restrict ourselves to the couplings of vector multiplets, for which the general couplings were found in [22, 23]. One can separate the ungauged part, and the part dependent on a gauge coupling constant  $g$ . We will consider only the gauging of a  $U(1)$   $R$ -symmetry group.

In step 2, the gauge coupling constant  $g$  is replaced by a field  $G(x)$  and a Lagrange multiplier field  $A^{(4)}$ , i.e.

$$g \longrightarrow G(x) \quad \text{plus} \quad A^{(4)}. \quad (55)$$

In step 3 we introduce the 3-brane action. That action has extra terms for the Lagrange multiplier  $A^{(4)}$ -form, which allows  $G(x)$  to vary crossing the 3-brane. We will show how every step preserves the supersymmetry.

Before embarking on this programme, we first review some basic properties of the D=5 bulk supersymmetry. We consider N=2 supersymmetry, i.e. 8 supercharges. These supercharges are part of the super-anti-de Sitter algebra  $SU(2, 2|1)$ . It involves the anti-de Sitter algebra  $SO(4, 2) \simeq SU(2, 2)$  with translations  $P_a$  and Lorentz rotations  $M_{ab}$ , the supersymmetries

$Q^i$ , with  $i = 1, 2$ , a symplectic Majorana spinor, and a  $U(1)$  generator as  $R$ -symmetry. The most characteristic (anti)commutator relations are

$$\begin{aligned}\{Q^i, Q^j\} &= \frac{1}{2}\varepsilon^{ij}\gamma_a P^a + igQ^{ij}\gamma^{ab}M_{ab} + i\varepsilon^{ij}U, \\ [U, Q^i] &= gQ^i_j Q^j, \\ [P_a, P_b] &= g^2 Q^i_j Q^j_i M_{ab}, \\ [P_a, Q^i] &= i\gamma_a g Q^i_j Q^j.\end{aligned}\tag{56}$$

The matrix  $Q_{ij}$  satisfies

$$\begin{aligned}Q_{ij} &= Q_{ji}, \quad Q^i_j \equiv \varepsilon^{ik}Q_{kj} = i(q_1\sigma_1 + q_2\sigma_2 + q_3\sigma_3), \\ q_1, q_2, q_3 &\in \mathbb{R}, \quad (q_1)^2 + (q_2)^2 + (q_3)^2 = 1.\end{aligned}\tag{57}$$

This matrix determines the embedding of  $U(1)$  in the automorphism group of the supersymmetries  $SU(2)$ . This choice is not physically relevant in itself. The second of the commutators in (56) implies that  $g$  is the coupling constant of  $R$ -symmetry. But the third equation says that  $g^2$  determines the curvature of spacetime, i.e. it determines the cosmological constant. This fact is the cornerstone of the situation that we describe. The gauge coupling and the cosmological constant are related.

We now consider the action of supergravity coupled to  $n$  vector multiplets [22, 23]. The fields are

$$e_\mu^\alpha, \psi_\mu^i, A_\mu^I, \varphi^x, \lambda^{ix},\tag{58}$$

i.e. the graviton, gravitini,  $n+1$  gauge fields ( $I = 0, 1, \dots, n$ ), including the graviphoton,  $n$  scalars ( $x = 1, \dots, n$ ), and  $n$  doublets of spinors. The scalars describe a manifold structure that has been called very special geometry [24]. That geometry, and the complete action, is determined by a symmetric tensor  $C_{IJK}$ . The scalars are best described as living in an  $n$ -dimensional scalar manifold embedded in an  $(n+1)$ -dimensional space. The  $h^I(\varphi)$  are the coordinates of this larger space. The submanifold is defined by an embedding condition such that the  $h^I$  as functions of the independent coordinates  $\varphi^x$  should satisfy

$$h^I(\varphi)h^J(\varphi)h^K(\varphi)C_{IJK} = 1.\tag{59}$$

The metric and all relevant quantities of the bulk theory only depend on  $C_{IJK}$ .

We now gauge a  $U(1)$  group. We take a linear combination of the vectors as gauge field for this  $R$ -symmetry. The linear combination is defined by real constants  $V_I$ :

$$A_\mu^{(R)} \equiv V_I A_\mu^I.\tag{60}$$

The action and the transformation laws are modified by terms that all depend on  $gQ^i_j$ .

### Step 1: $\mathbb{Z}_2$ -symmetries

We need a  $\mathbb{Z}_2$ -symmetry that involves an inversion of the transverse direction, i.e.  $x_5 \rightarrow -x_5$  and that satisfies the property

$$G(x) : \text{odd} \qquad A^{(4)} : \text{even}.\tag{61}$$



Like in D=10 the field equation of  $A^{(4)}$  imposes the constancy of  $G(x)$  such that effectively it is a constant.

There indeed exists a  $\mathbb{Z}_2$ -symmetry with the desired properties. Its action on the fünf-bein is given by

$$\begin{aligned}\Pi(e_\mu^m) &= +1, & \Pi(e_\mu^5) &= -1, \\ \Pi(e_5^m) &= -1, & \Pi(e_5^5) &= +1.\end{aligned}\tag{62}$$

For the action on all the other fields, see [2]. In particular, its action on the fermions involves a  $2 \times 2$  matrix  $M^i_j$ :

$$\lambda^i(x^5) \longrightarrow \Pi(\lambda)\gamma_5 M^i_j \lambda^j(-x^5),\tag{63}$$

where  $\lambda$  is a symplectic Majorana spinor. For instance, the action on the gravitino is given by

$$\Pi(\psi_\mu) = +1, \quad \Pi(\psi_5) = -1.\tag{64}$$

The above  $\mathbb{Z}_2$ -symmetry is an exact symmetry for  $g = 0$ . For  $g \neq 0$  we find the following formal relationship:

$$MQ = -\Pi(g)QM.\tag{65}$$

We distinguish the following two cases:

- $\Pi(g) = +1$  ,  $QM = -MQ$  .
- $\Pi(g) = -1$  ,  $M \sim Q$  .

We see that the relative choice of  $M$  and the  $Q$  in (57) matters. If they anticommute, the choice that has been taken in [5, 6], then  $g$  does not change when one crosses the brane. If they commute, as in [7, 8], then  $g$  jumps over the brane. We will consider the second case.

We distinguish between even and odd fields. The circle condition on the fields and the orbifold condition are then

$$\begin{aligned}\Phi(x^5) &= \Phi(x^5 + 2\tilde{x}^5), \\ \Phi_{\text{even}}(-x^5) &= \Phi_{\text{even}}(x^5), \quad \Phi_{\text{odd}}(-x^5) = -\Phi_{\text{odd}}(x^5).\end{aligned}\tag{66}$$

These conditions imply that odd fields vanish on the branes, i.e. at  $x^5 = 0$  and at  $x^5 = \tilde{x}^5$ . Also the supersymmetries split. Half of them are even, and half are odd. Therefore, on the brane one has 4 supersymmetries, i.e.  $\mathcal{N} = 1$  in 4 dimensions.

## Step 2: rewriting D=5 matter coupled supergravity

Our aim is to replace the coupling constant  $g$  by a coupling field  $G(x)$ . In the Günaydin–Sierra–Townsend (GST) action, the coupling constant appears up to terms in  $g^2$ . We thus replace

$$S_{GST}(g) = S_0 + gS_1 + g^2S_2 \Rightarrow S_{GST}(G(x)) = S_0 + G(x)S_1 + G(x)^2S_2.\tag{67}$$

Another term is added to the bulk action that forces  $G(x)$  to be a constant, using a Lagrange-multiplier 4-form  $A_{\mu\nu\rho\sigma}$ :

$$\begin{aligned}
S_{\text{bulk}} &= S_{GST}(G(x)) + \int d^5x e \frac{1}{4!} \varepsilon^{\mu\nu\rho\sigma\tau} A_{\mu\nu\rho\sigma} \partial_\tau G(x) \\
&= S_0 - \int d^5x e V - \int d^5x e \hat{F}(x) G(x) + \text{fermionic terms.}
\end{aligned} \quad (68)$$

In the second line, the terms have been reordered. The potential  $V$  originates from  $S_2$  in (67), and leads to the potential

$$V = -6G^2 \left[ W^2 - \frac{3}{4} \left( \frac{\partial W}{\partial \varphi^x} \right)^2 \right], \quad W \equiv \sqrt{\frac{2}{3}} h^I V_I, \quad (69)$$

where the linear combination  $W$  appears, analogous to (60). The third term in (68) appears from integrating by part the term with the Lagrange multiplier, leading to the flux

$$\hat{F} \equiv \frac{1}{4!} e^{-1} \varepsilon^{\mu\nu\rho\sigma\tau} \partial_\mu A_{\nu\rho\sigma\tau} + \text{covariantization.} \quad (70)$$

The covariantization terms come from  $S_1$  in (67). This method of describing a constant using a 4-form is in fact an old method that was already used in [25].

It is easy to understand how supersymmetry is preserved. Indeed, the GST action is known to be invariant:

$$\delta(\epsilon) S_{GST}(g) = 0. \quad (71)$$

Therefore, the only non-invariance for  $S_{GST}(G(x))$  appears, if we define  $\delta(\epsilon)G = 0$ , from the  $x$ -dependence of  $G(x)$ . It is thus proportional to its spacetime derivative

$$\delta(\epsilon) S_{GST}(G(x)) = B^\mu \partial_\mu G(x), \quad (72)$$

where  $B^\mu$  is some expression of the other fields and parameters, whose exact form is not important for the argument here. One immediately sees then that invariance of (68) is obtained by defining the transformation law of the 4-form as

$$\begin{aligned}
\delta(\epsilon) \frac{1}{4!} \varepsilon^{\mu\nu\rho\sigma\tau} A_{\mu\nu\rho\sigma} = B^\tau = e \left[ -i \frac{3}{2} \bar{\psi}_\mu^i \gamma^{\mu\tau} \epsilon^j W - \bar{\psi}_\mu^i \gamma^{\mu\tau\rho} \epsilon^j A_\rho^{(R)} \right. \\
\left. + \frac{3}{2} \bar{\lambda}_x^i W^{,x} \gamma^\tau \epsilon^j \right] Q_{ij}.
\end{aligned} \quad (73)$$

### Step 3: adding the 3-brane actions

As the last step we introduce the brane action, such that the total action is

$$S_{\text{new}} = S_{\text{bulk}} + S_{\text{brane}}. \quad (74)$$

The brane action has the form

$$\begin{aligned}
S_{\text{brane}} &= -2g \int d^5x \left( \delta(x^5) - \delta(x^5 - \tilde{x}^5) \right) \left( e_{(4)} 3W + \frac{1}{4!} \varepsilon^{\underline{\mu}\underline{\nu}\underline{\rho}\underline{\sigma}} A_{\underline{\mu}\underline{\nu}\underline{\rho}\underline{\sigma}} \right) \\
&= S_{\text{brane},1} - S_{\text{brane},2}.
\end{aligned} \quad (75)$$

Underlined indices refer to the values in the brane directions:  $\underline{\mu} = 0, 1, 2, 3$ . The action is presented as an integral over 5 dimensions, but the delta functions imply that it is a four-dimensional action for each brane separately. The action of each brane consists of a Dirac–Born–Infeld (DBI) term and a Wess–Zumino (WZ) term. However, both parts depend only on the pullback of the bulk fields to the branes. There are no fields living on the brane. The function  $W$  appears in the DBI term, and plays the role of the central charge of the brane. But most importantly, the 4-form Lagrange multiplier appears in the WZ term, and this thus modifies its field equation. The new field equation is

$$\partial_5 G(x^5) = 2g (\delta(x^5) - \delta(x^5 - \tilde{x}^5)) , \quad (76)$$

and leads to the solution (taking into account the cyclicity condition)

$$G(x) = g \varepsilon(x^5) . \quad (77)$$

The function  $\varepsilon(x^5)$  jumps as well at  $x^5 = 0$  as at  $x^5 = \tilde{x}^5$ , see Fig. 2. It is clear from this picture that we need the second brane. Indeed, one has to come back to the original value of  $g$ , in order that total derivatives in  $x^5$  do not contribute to the action. The flux, which is determined by the field equation of  $G(x)$ , is

$$\hat{F} = 12G \left[ W^2 - \frac{3}{4} \left( \frac{\partial W}{\partial \varphi^x} \right)^2 \right] + \text{fermionic terms} . \quad (78)$$

The overall factor changes when crossing each brane due to (77). These jumps imply that the wall acts as a sink for the fluxes.

That supersymmetry is still preserved by the addition of the brane is less obvious and is the non-trivial part of the construction. It turns out that the supersymmetry is preserved thanks to the projections. One finds (indices  $m$  are tangent space indices in brane directions)

$$\begin{aligned} \delta S_{\text{brane}} = & -3g \int d^5 x (\delta(x^5) - \delta(x^5 - \tilde{x}^5)) e_{(4)} \left[ W \bar{\epsilon}^i \gamma^m e_m^\mu (\psi_{\underline{\mu}i} - i \gamma_5 Q_{ij} \psi_{\underline{\mu}}^j) + \right. \\ & \left. + W_{,x} \bar{\epsilon}^i (i \lambda_i^x - \gamma_5 Q_{ij} \lambda^{xj}) \right] . \end{aligned} \quad (79)$$

The combinations of the gravitino and the gauginos that are in brackets are the components that are odd under the  $\mathbb{Z}_2$  projection, and thus vanish on the brane. This leads to the invariance. Remark that in each case one of the two terms comes from the DBI (mass) term and the other from the WZ (charge) term. This therefore determines the relative weight of the two terms, and is the mass = charge relation, that says that the brane is BPS. We thus see, indeed, that the brane action is separately invariant. Note, that if we would not use (or eliminate) the Lagrange multiplier, then this would relate bulk and brane, and only the sum would be invariant.

### A smooth Randall-Sundrum scenario

The RS scenario in 5 dimensions can be made supersymmetric despite the singularities of the space. The action and transformation laws can be obtained using a 4-form, such that bulk and

brane are separately supersymmetric. Supersymmetric solutions exist with fixed scalars or 1/2 supersymmetry [2]. Half of the supersymmetries vanish on the branes. Also the translation generator in the fifth direction vanishes on the brane. That is how the algebra can be realized.

What is less clear is whether a *smooth* RS scenario exists. In that case the aim is to find a smooth, i.e. without  $\delta$ -function source terms, domain wall solution of the form

$$ds^2 = a^2(x^5) dx^\mu dx^\nu \eta_{\mu\nu} + (dx^5)^2, \quad (80)$$

where  $a(x^5)$  is a so-called warp factor with

$$a(x^5) \rightarrow 0 \quad \text{for} \quad x^5 \rightarrow \pm\infty. \quad (81)$$

In terms of the AdS/CFT language, where the warp factor plays the role of an energy scale, it means that we need a renormalization group flow from an infrared to an infrared point.

Whether or not it is possible to find a smooth domain wall solution depends crucially on the detailed properties of the scalar potential. Recently, some progress has been made involving the inclusion of hyper multiplets [26] (see also [27]) but a fully satisfactory solution has not been found yet. On the other hand there is no *a priori* obstruction for such a solution to exist. In view of this it is clearly important to construct the most general matter coupled supergravity theory in D=5 and hence obtain the most general scalar potential. In view of this we started to attack this problem using the conformal approach. We will finish this lecture by presenting a short review of the preliminary results we have obtained sofar. For more details, see [28, 29]

#### 4 The conformal approach

To explain the conformal approach it is instructive to consider a single metric field  $g_{\mu\nu}$  in D=4. Under general coordinate transformations this field transforms as

$$\delta_{\text{g.c.t.}} g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \quad (82)$$

Due to these gauge transformations the  $\mu$ -index has effectively 4-1=3 values and we count the off-shell degrees of freedom as

$$1/2 \times 3 \times 4 = 6 = \underbrace{5}_{\text{spin } 2} + \underbrace{1}_{\text{spin } 0}. \quad (83)$$

Now consider an extra gauge transformation under dilatations:

$$\delta g_{\mu\nu} = -2\Lambda_D g_{\mu\nu}. \quad (84)$$

Any theory containing a metric field  $g_{\mu\nu}$  can be made invariant under dilatations by using a compensating field  $\phi$  which transforms under dilatations as

$$\delta\phi = \Lambda_D \phi. \quad (85)$$

For instance the Poincaré Lagrangean

$$\mathcal{L}_{\text{Poinc.}} = -\frac{1}{12} \sqrt{g} R \quad (86)$$

can be converted into a dilatation invariant Lagrangean

$$\mathcal{L}_{\text{conf.}} = \sqrt{g} \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} R \phi^2 \right) \quad (87)$$

by use of the compensating field  $\phi$ . Note that for the gauge choice  $\phi = 1$  the conformal Lagrangean (87) reduces to the Poincaré Lagrangean (86).

The advantages of rewriting a supergravity Lagrangean in a conform invariant way and then considering general (conformal) matter couplings are numerous. For instance

- In the conformal approach the basic building blocks are smaller and have more symmetries.
- In the conformal approach one has a manifest duality structure.
- In the process one learns interesting things about the geometry of the scalar manifolds involved
- Hopefully it will lead to more general matter couplings.

We first consider the algebra of bosonic conformal transformations. Given a spacetime with a metric tensor  $g_{\mu\nu}(x)$  the conformal transformations are defined as the general coordinate transformations that leaves “angles” invariant. The parameters of these special coordinate transformations define a conformal Killing vector  $k^\mu(x)$ . The defining equation for this conformal Killing vector is given by

$$\delta_{\text{g.c.t.}}(k)g_{\mu\nu}(x) \equiv \nabla_\mu k_\nu(x) + \nabla_\nu k_\mu(x) = \omega(x)g_{\mu\nu}(x), \quad (88)$$

where  $\omega(x)$  is an arbitrary function,  $k_\mu = g_{\mu\nu}k^\nu$  and the covariant derivative is given by  $\nabla_\mu k_\nu = \partial_\mu k_\nu - \Gamma_{\mu\nu}^\rho k_\rho$ . In flat  $D$ -dimensional Minkowski spacetime, (88) implies

$$\partial_{(\mu} k_{\nu)}(x) - \frac{1}{D} \eta_{\mu\nu} \partial_\rho k^\rho(x) = 0. \quad (89)$$

In  $D=5$  the conformal algebra is finite-dimensional. The solutions of (89) are given by

$$k^\mu(x) = \xi^\mu + \lambda_M^{\mu\nu} x_\nu + \lambda_D x^\mu + (x^2 \Lambda_K^\mu - 2x^\mu x \cdot \Lambda_K). \quad (90)$$

Corresponding to the parameters  $\xi^\mu$  are the translations  $P_\mu$ , the parameters  $\lambda_M^{\mu\nu}$  correspond to Lorentz rotations  $M_{\mu\nu}$ , to  $\lambda_D$  are associated the dilatations  $D$ , and  $\Lambda_K^\mu$  are the parameters of ‘special conformal transformations’  $K_\mu$ . Thus, the full set of conformal transformations  $\delta_C$  can be expressed as follows:

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu} M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu. \quad (91)$$

The commutators between the different generators define the conformal algebra which, for  $D=5$ , is isomorphic to the algebra  $SO(5, 2)$ .

We next consider the extension to conformal supersymmetry. The additional supersymmetries are defined as the supersymmetries that leave the gamma-traceless part of the gravitino invariant. The parameters of these supersymmetries define a conformal Killing spinor  $\epsilon^i(x)$  whose defining equation is given by

$$\delta_Q(\epsilon)\psi_\mu^i \equiv \nabla_\mu \epsilon^i(x) = \lambda(x)\gamma_\mu \gamma^\nu \psi_\nu^i. \quad (92)$$

In  $D$ -dimensional Minkowski spacetime this equation implies

$$\partial_\mu \epsilon^i - \frac{1}{D} \gamma_\mu \gamma^\nu \partial_\nu \epsilon^i = 0. \quad (93)$$

The solution to this equation is given by

$$\epsilon^i(x) = \epsilon^i + i x^\mu \gamma_\mu \eta^i, \quad (94)$$

where the (constant) parameters  $\epsilon^i$  correspond to “ordinary” supersymmetry transformations  $Q_\alpha^i$  and the parameters  $\eta^i$  define special conformal supersymmetries generated by  $S_\alpha^i$ . The conformal transformation (90) and the supersymmetries (94) do not form a closed algebra. To obtain closure one must introduce additional automorphism-symmetry generators. In particular, in the case of 8 supercharges  $Q_\alpha^i$  in  $D = 5$ , there is an additional  $SU(2)$  automorphism-symmetry with generators  $U_{ij} = U_{ji}$  ( $i = 1, 2$ ). Thus, the full set of superconformal transformations  $\delta_C$  is given by:

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu} M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu + \lambda^{ij} U_{ij} + i\bar{\epsilon} Q + i\bar{\eta} S. \quad (95)$$

We refer to [28] for the full superconformal algebra  $F^2(4)$  formed by (anti-)commutators between the (bosonic and fermionic) generators. Note that the superconformal algebra contains as a bosonic subalgebra:

$$F^2(4) \supset SO(5, 2) \times SU(2). \quad (96)$$

We give below the  $Q$  and  $S$  commutators of the (rigid) superconformal algebra expressed in terms of commutators of variations of the fields. These commutators are realized on the Weyl multiplet and *all* matter multiplets:

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P\left(\frac{1}{2}\bar{\epsilon}_2\gamma_\mu\epsilon_1\right), \quad (97)$$

$$[\delta_S(\eta), \delta_Q(\epsilon)] = \delta_D\left(\frac{1}{2}i\bar{\epsilon}\eta\right) + \delta_M\left(\frac{1}{2}i\bar{\epsilon}\gamma^{ab}\eta\right) + \delta_U\left(-\frac{3}{2}i\bar{\epsilon}^{(i}\eta^{j)}\right), \quad (98)$$

$$[\delta_S(\eta_1), \delta_S(\eta_2)] = \delta_K\left(\frac{1}{2}\bar{\eta}_2\gamma^a\eta_1\right). \quad (99)$$

We now briefly review the conformal programme in 4 steps. For more details, see [28, 29].

### Step 1: the Weyl multiplet

In a first step we construct conformal supergravity as the gauge theory of the superconformal algebra  $F^2(4)$ . We apply the methods developed first for  $N = 1$  in 4 dimensions [30]. They are based on gauging the conformal superalgebra [31], which, in our case, is  $F^2(4)$ . The generators of the superconformal algebra  $F^2(4)$  are listed in Table 2. We only discuss the general method. For full details I refer to [28].

We assign to every generator of the superconformal algebra a gauge field. These gauge fields and the names of the corresponding gauge parameters are given in Table 2. It turns out that in the process of gauging some of these gauge fields remain *independent*, like the fünfbein  $e_\mu{}^a$  while others become *dependent* like the spin connection field  $\omega_\mu{}^{ab}$ :

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e, b, \psi_\mu). \quad (100)$$

Table 2. The gauge fields and parameters of the superconformal algebra  $F^2(4)$

Generators	$P_a$	$M_{ab}$	$D$	$K_a$	$U_{ij}$	$Q_{\alpha i}$	$S_{\alpha i}$
Fields	$e_\mu^a$	$\omega_\mu^{ab}$	$b_\mu$	$f_\mu^a$	$V_\mu^{ij}$	$\psi_\mu^i$	$\phi_\mu^i$
Parameters	$\xi^a$	$\lambda^{ab}$	$\Lambda_D$	$\Lambda_K^a$	$\Lambda^{ij}$	$\epsilon^i$	$\eta^i$

This is not yet enough. In order to get the correct counting of degrees of freedom one must introduce additional matter fields. It turns out that at this point there are two different ways of introducing matter fields leading to two different Weyl multiplets: the “standard” Weyl multiplet and the “dilaton” Weyl multiplet. The fact that there exist two different versions of conformal supergravity has been encountered before in 6 dimensions [32]. Table 3 suggests that the same feature might also occur in 4 dimensions.

Table 3. The two different formulations of the Weyl multiplet in  $D = 4, 5, 6$

Dimension $D$	# d.o.f.	Standard Weyl	Dilaton Weyl
6	10	$T_{abc}^+$	$B_{\mu\nu}$
5	10	$T_{ab}$	$A_\mu, B_{\mu\nu}$
4	6	$T_{ab}$	$A_\mu, B_\mu$

It seems plausible that in this case the coupling of a vector multiplet to the Standard Weyl multiplet will give a dilaton Weyl multiplet containing two vectors. It would be interesting to see whether the dilaton Weyl multiplet in 4 dimensions indeed exists, and whether it may lead to new matter couplings. D=5 matter couplings related to the dilaton Weyl multiplet have recently been considered in [33].

Step 2: matter multiplets

As a second step we introduce matter multiplets and couple them to conformal supergravity. Below we discuss the different matter multiplets.

• vector multiplets

The component fields of the D=5 off-shell 8+8 vector multiplet in the adjoint representation are given by

$$\left\{ \underbrace{A_\mu}_{\text{gauge symmetries}}, \underbrace{\psi^i}_{\text{gaugino}}, \underbrace{\sigma}_{\text{scalar}}, \underbrace{Y^{ij}}_{\text{auxiliary field}} \right\}. \tag{101}$$

We have indicated the  $SU(2)$  labels and Weyl weights in Table 4.

The supersymmetry transformations of the fields can be found in [29]. The superconformal algebra (99) is modified with additional gauge transformations, i.e.

$$\text{vector algebra} = \text{superconformal algebra} + \text{gauge symmetries}. \tag{102}$$

Table 4. The  $8 + 8$  off-shell vector multiplet

Field	$SU(2)$	$w$	# d.o.f.
$A_\mu$	1	0	4
$Y^{ij}$	3	2	3
$\sigma$	1	1	1
$\psi^i$	2	$3/2$	8

In particular we have

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_P\left(\frac{1}{2}\bar{\epsilon}_2\gamma_\mu\epsilon_1\right) + \delta_G\left(-\frac{1}{2}i\sigma\bar{\epsilon}_2\epsilon_1\right). \quad (103)$$

Note that even though we are considering rigid superconformal symmetry, the algebra (103) contains a field-dependent term on the righthand side. Such soft terms are commonplace in local superconformal symmetry but here they already appear at the rigid level. In Hamiltonian language it means that the algebra is satisfied modulo constraints.

#### • vector-tensor multiplets

From work on  $N = 2, D = 5$  Poincaré matter couplings [34] it is known that vector multiplets transforming in representations other than the adjoint have to be dualized to tensor fields. We define a vector-tensor multiplet to be a vector multiplet transforming in a reducible representation that contains the adjoint representation as well as another, arbitrary representation.

The vector-tensor multiplet contains *a priori* an arbitrary number of tensor fields. The restriction to an even number of tensor fields is not imposed by the closure of the algebra. If one demands that the field equations do not contain tachyonic modes an even number is required [35]. Closely related to this is the fact that one can only construct an action for an even number of tensor multiplets. But supersymmetry without an action allows the more general possibility. Note that these results are independent of the use of superconformal or super-Poincaré algebras.

Starting from  $n + 1$  vector multiplets we obtain a vector-tensor multiplet by considering a more general set of fields  $\mathcal{H}_{\mu\nu}^{\tilde{I}}$  ( $\tilde{I} = 0, \dots, n + m$ ). We write  $\mathcal{H}_{\mu\nu}^{\tilde{I}} = \{F_{\mu\nu}^I, B_{\mu\nu}^M\}$  with  $\tilde{I} = (I, M)$  ( $I = 0, \dots, n; M = n + 1, \dots, n + m$ ). The first part of these fields correspond to the generators in the adjoint representation. These are the fields belonging to the off-shell vector multiplet (see above). The other fields may belong to an arbitrary, possibly reducible, representation. It turns out that they belong to an on-shell tensor multiplet. For more details, we refer to [29].

#### • hyper multiplet

As for the tensor multiplets, there is in general no known off shell formulation with a finite number of auxiliary fields. Therefore already the algebra leads to equations of motion.



A single hypermultiplet contains four real scalars and two spinors subject to the symplectic Majorana reality condition. For  $r$  hypermultiplets, we introduce real scalars  $q^X(x)$ , with  $X = 1, \dots, 4r$ , and spinors  $\zeta^A(x)$  with  $A = 1, \dots, 2r$ :

$$\left\{ \underbrace{q^X}_{\text{scalars}}, \underbrace{\zeta^A}_{\text{spinors}} \right\}. \quad (104)$$

For supersymmetry transformations (with  $\epsilon^i$  constant parameters) of the bosons  $q^X(x)$ , we allow some general form parametrized by arbitrary functions  $f_{iA}^X(q)$ . Also for the transformation rules of the fermions we write the general form compatible with the supersymmetry algebra. This introduces other general functions  $f_X^{iA}(q)$  and  $\omega_{XB}{}^A(q)$ :

$$\begin{aligned} \delta(\epsilon)q^X &= -i\epsilon^i \zeta^A f_{iA}^X, \\ \delta(\epsilon)\zeta^A &= \frac{1}{2}i\gamma^\mu(\partial_\mu q^X)f_X^{iA}\epsilon_i - \zeta^B \omega_{XB}{}^A(\delta(\epsilon)q^X). \end{aligned} \quad (105)$$

The functions satisfy reality properties consistent with reality of  $q^X$  and the symplectic Majorana conditions on  $\zeta^A$ .

A priori the functions  $f_{iA}^X$  and  $f_X^{iA}$  are independent, but the commutator of two supersymmetries on the scalars only gives a translation if one imposes

$$\begin{aligned} f_Y^{iA} f_{iA}^X &= \delta_Y^X, & f_X^{iA} f_{jB}^X &= \delta_j^i \delta_B^A, \\ \mathcal{D}_Y f_{iB}^X &\equiv \partial_Y f_{iB}^X - \omega_{YB}{}^A f_{iA}^X + \Gamma_{ZY}{}^X f_{iB}^Z = 0. \end{aligned} \quad (106)$$

where  $\Gamma_{XY}{}^Z$  is some object, symmetric in the lower indices. This means that  $f_{iA}^X$  and  $f_X^{iA}$  are each others inverse and are covariantly constant with respect to reparametrizations of the scalars and fermions with connections  $\Gamma$  and  $\omega$  respectively.

The conditions (106) encode all the constraints on the target space that follow from imposing the supersymmetry algebra. There are no further geometrical constraints coming from the fermion commutator; instead this commutator defines the equations of motion for the on-shell hypermultiplet.

The geometry of the target space is that of a *hypercomplex* manifold. It is a weakened version of hyperkähler geometry where no metric is available. But there is a triplet of complex structures (the hypercomplex), defined as

$$J_X{}^{Y\alpha} \equiv -if_X^{iA}(\sigma^\alpha)_i{}^j f_{jA}^Y. \quad (107)$$

Using (106), they are covariantly constant and satisfy the quaternion algebra

$$J^\alpha J^\beta = -I_{4r} \delta^{\alpha\beta} + \varepsilon^{\alpha\beta\gamma} J^\gamma. \quad (108)$$

As far as the superconformal supersymmetries are concerned we only mention the dilatations. For the scalar fields, it is convenient to consider the set of fields  $q^X$  as the coordinates of a scalar manifold with affine connection  $\Gamma_{XY}{}^Z$ . With this understanding the transformation of  $q^X$  under dilatations can be characterized by:

$$\delta_\Delta(\lambda_D)q^X = \lambda_D k^X(q). \quad (109)$$

The commutator (98) only closes if we impose

$$\mathcal{D}_Y k^X \equiv \partial_Y k^X + \Gamma_{YZ}^X k^Z = \frac{3}{2} \delta_Y^X. \quad (110)$$

Finally, we may have symmetries on the hypercomplex manifold. Gauging these symmetries leads to additional terms in the equations of motion. For more details, see [29].

• *linear multiplet*

The off-shell 8+8 linear multiplet contains a constrained vector  $E_a$ . In [36] a version of the linear multiplet containing a 4-form gauge field  $A_{\mu\nu\rho\sigma}$  has been discussed. This gauge field could be used to explain the conformal origin of the 4-form Lagrange multiplier field introduced in [2].

### Step 3: the action

As a third step one must construct rigid superconformal actions for all matter multiplets discussed so far and couple them to conformal supergravity. These results are given in [29]. In general the existence of an action is more restrictive than only considering equations of motion. For instance, the construction of an action for the vector multiplet or the vector-tensor multiplet, requires the existence of a symmetric tensor

$$C_{IJK}. \quad (111)$$

This symmetric tensor determines all the couplings and defines a very special real geometry.

On the other hand the construction of an action for the hypermultiplets requires the existence of a metric on the scalar manifold:

$$g_{XY} = f_X^{iA} C_{AB} \varepsilon_{ij} f_Y^{jB}, \quad (112)$$

where  $C_{AB}$  can be chosen to be constant. The tensor  $C_{AB}$  is a  $USp(2r)$  metric that can be used to raise and lower the  $A$ -indices. The relation between the action and the equations of motion is given by

$$\frac{\delta S}{\delta \bar{\zeta}^A} = 2C_{AB} \Gamma^B, \quad (113)$$

where  $\Gamma^B$  are the nonclosure functions.

The necessity of a metric makes us conclude that we can only write down a hyper multiplet action for a Hyperkähler scalar manifold.

### Step 4: gauge-fixing

For the gauge-fixing one needs at least one compensating vector multiplet and one compensating hyper multiplet. The different fields are used for gauge-fixing the following conformal symmetries:

$$\left\{ \underbrace{A_\mu}_{\text{Poinc. SUGRA}}, \underbrace{\psi}_S, \underbrace{\sigma}_D \right\}. \quad (114)$$

$$\left\{ \underbrace{\quad}_{\text{Poinc. SUGRA}}, \underbrace{\quad}_{\text{SU}(2) + D} \right\}. \quad (115)$$

The fields  $A_\mu$  and  $\zeta$  become part of the Poincaré supergravity multiplet. This should lead to a general Poincaré matter coupled supergravity theory. We hope to publish more about the details in the nearby future.

## 5 Summary

In this lecture I have described how supersymmetry can be introduced in singular spaces. For the D=10 case this describes the Type I' string theory while for D=5 we are dealing with the RS scenario including sources. Our construction does not apply to D=11 since the D=11 end of the world branes are not BPS objects with mass = charge. In fact, they carry no charge whatsoever.

The question remains open whether a smooth RS scenario exists. To answer this question more definitely it is of crucial importance that we know the most general D=5 matter coupled supergravity. Although there is already literature about this we decided, based on experiences in the past, to use the conformal approach to construct such matter couplings. I outlined the different steps involved in this programme. More definite results are expected to appear shortly [29]. It would be very interesting to see whether possible generalized matter couplings will allow for a smooth RS scenario.

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